



TITLE:

# CONVOLUTION OF CERTAIN STARLIKE FUNCTIONS WITH NEGATIVE COEFFICIENTS

AUTHOR(S):

AOUF, M.K.; NUNOKAWA, M.

---

CITATION:

AOUF, M.K. ...[et al]. CONVOLUTION OF CERTAIN STARLIKE FUNCTIONS WITH NEGATIVE COEFFICIENTS. 数理解析研究所講究録 1994, 881: 36-47

ISSUE DATE:

1994-08

URL:

<http://hdl.handle.net/2433/84225>

RIGHT:

## CONVOLUTION OF CERTAIN STARLIKE FUNCTIONS WITH NEGATIVE COEFFICIENTS

M. K. AOUF(Mansoura大学) and M. NUNOKAWA(布川 護, 群馬大学)

**ABSTRACT.** The object of the present paper is to determine several interesting results for the modified Hadamard product (convolution) of functions belonging to the class  $T_{\lambda,n}^*(A,B)$  consisting of regular functions with negative coefficients.

---

**KEY WORDS-** Regular, Starlike, Ruscheweyh derivative, convolution.

AMS (1991) Subject Classification. 30C45.

## 1. Introduction

Let  $T$  denote the class of functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \quad (a_k \geq 0) \quad (1.1)$$

which are regular in the unit disc  $U = \{z: |z| < 1\}$ . If  $f(z)$  defined by (1.1) and  $g(z)$  defined by

$$g(z) = z - \sum_{k=2}^{\infty} b_k z^k \quad (b_k \geq 0). \quad (1.2)$$

The modified Hadamard product (convolution) of  $f(z)$  and  $g(z)$  is defined by the power series

$$f * g(z) = z - \sum_{k=2}^{\infty} a_k b_k z^k. \quad (1.3)$$

The  $n$ -th order Ruscheweyh derivative of  $f(z)$ , denoted by  $D^n f(z)$ , is defined by

$$D^n f(z) = \frac{z(z^{n-1} f(z))^{(n)}}{n!}, \quad (1.4)$$

where  $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ ,  $\mathbb{N} = \{1, 2, \dots\}$ . Ruscheweyh [3] determined that

$$D^n f(z) = \frac{z}{(1-z)^{n+1}} * f(z) \quad (n \in \mathbb{N}_0). \quad (1.5)$$

It is easy to see that

$$D^n f(z) = z - \sum_{k=2}^{\infty} \delta(n,k) a_k z^k, \quad (1.6)$$

where

$$\delta(n,k) = \begin{bmatrix} n+k-1 \\ n \end{bmatrix}. \quad (1.7)$$

A function  $f(z) \in T$  is said to be in the class  $T_{\lambda,n}^*(A,B)$  if it satisfies the following condition

$$\left| \frac{\frac{D^{n+1}f(z)}{D^n f(z)} - 1}{\lambda(A-B) - B \left[ \frac{D^{n+1}f(z)}{D^n f(z)} - 1 \right]} \right| < 1, \quad z \in U, \quad (1.8)$$

where  $0 < \lambda \leq 1$ ,  $-1 \leq B < A \leq 1$ , and  $-1 \leq B \leq 0$ . The class  $T_{\lambda,n}^*(A,B)$  was studied by Kumar and Chaudhary [1].

In order to prove our results for functions belonging to the class  $T_{\lambda,n}^*(A,B)$ , we shall require the following lemma given by Kumar and Chaudhary [1].

**LEMMA 1.** Let the function  $f(z)$  defined by (1.1). Then  $f(z) \in T_{\lambda,n}^*(A,B)$  if and only if

$$\sum_{k=2}^{\infty} C_{k,\lambda} \delta(n,k) a_k \leq \lambda(A-B)(n+1), \quad (1.9)$$

where

$$C_{k,\lambda} = (1-B)(k-1) + \lambda(A-B)(n+1). \quad (1.10)$$

The result is sharp.

## 2. Modified Hadamard Product

**THEOREM 1.** Let the function  $f(z)$  defined by (1.1) and  $g(z)$  defined by (1.2) be in the class  $T_{\lambda,n}^*(A,B)$ . Then  $f * g(z) \in T_{\mu,n}^*(A,B)$ , where

$$\mu = \frac{\lambda^2(A-B)(1-B)}{C_{2,\lambda}^2 - \lambda^2(A-B)^2(n+1)}. \quad (2.1)$$

The result is sharp.

**PROOF.** Employing the technique used earlier by Schild and Silverman [2]. We need to find the largest  $\mu$  such that

$$\sum_{k=2}^{\infty} \frac{C_{k,\mu} \delta(n,k)}{\mu(A-B)(n+1)} a_k b_k \leq 1. \quad (2.2)$$

Since

$$\sum_{k=2}^{\infty} \frac{C_{k,\lambda} \delta(n,k)}{\lambda(A-B)(n+1)} a_k \leq 1 \quad (2.3)$$

and

$$\sum_{k=2}^{\infty} \frac{C_{k,\lambda} \delta(n,k)}{\lambda(A-B)(n+1)} b_k \leq 1, \quad (2.4)$$

by the Cauchy-Schwarz inequality we have

$$\sum_{k=2}^{\infty} \frac{C_{k,\lambda} \delta(n,k)}{\lambda(A-B)(n+1)} \sqrt{a_k b_k} \leq 1. \quad (2.5)$$

Thus it is sufficient to show that

$$\frac{C_{k,\mu} \delta(n,k)}{\mu(A-B)(n+1)} a_k b_k \leq \frac{C_{k,\lambda} \delta(n,k)}{\lambda(A-B)(n+1)} \sqrt{a_k b_k} \quad (k \geq 2), \quad (2.6)$$

that is, that

$$\sqrt{a_k b_k} \leq \frac{\mu C_{k,\lambda}}{\lambda C_{k,\mu}}. \quad (2.7)$$

Note that

$$\sqrt{a_k b_k} \leq \frac{\lambda(A-B)(n+1)}{C_{k,\lambda} \delta(n,k)} \quad (k \geq 2). \quad (2.8)$$

Consequently, we need only to prove that

$$\frac{\lambda(A-B)(n+1)}{C_{k,\lambda} \delta(n,k)} \leq \frac{\mu C_{k,\lambda}}{\lambda C_{k,\mu}} \quad (k \geq 2). \quad (2.9)$$

or, equivalently that,

$$\mu \leq \frac{\lambda^2 (A-B) (1-B) (n+1) (k-1)}{C_{k,\lambda}^2 \delta(n,k) - \lambda^2 (A-B)^2 (n+1)^2} \quad (2.10)$$

Since

$$D(k) = \frac{\lambda^2 (A-B) (1-B) (n+1) (k-1)}{C_{k,\lambda}^2 \delta(n,k) - \lambda^2 (A-B)^2 (n+1)^2} \quad (2.11)$$

is an increasing function of  $k$  ( $k \geq 2$ ), letting  $k = 2$  in (2.11) we obtain

$$\mu \leq D(2) = \frac{\lambda^2 (A-B) (1-B)}{C_{2,\lambda}^2 - \lambda^2 (A-B)^2 (n+1)}, \quad (2.12)$$

which completes the proof of Theorem 1.

Finally, by taking the functions

$$f(z) = g(z) = z - \frac{\lambda (A-B)}{C_{2,\lambda}} z^2 \quad (2.13)$$

we can see that the result in Theorem 1 is sharp.

**COROLLARY 1.** For  $f(z)$  and  $g(z)$  as in Theorem 1, we have

$$h(z) = z - \sum_{k=2}^{\infty} \sqrt{a_k b_k} z^k \quad (2.14)$$

belongs to the class  $T_{\lambda,n}^*(A,B)$ .

The result follows from the inequality (2.5). It is sharp for the same functions as in Theorem 1.

**THEOREM 2.** Let the function  $f(z)$  defined by (1.1) be in the class  $T_{\lambda,n}^*(A,B)$  and the function  $g(z)$  defined by (1.2) be in the class  $T_{\beta,n}^*(A,B)$ , then  $f * g(z) \in T_{\zeta,n}^*(A,B)$ , where

$$\zeta = \frac{\lambda\beta(A-B)(1-B)}{C_{2,\lambda}C_{2,\beta} - \lambda\beta(A-B)^2(n+1)}. \quad (2.15)$$

The result is sharp.

**PROOF.** Proceeding as in the proof of Theorem 1, we get

$$\zeta \leq E(k) = \frac{\lambda\beta(A-B)(1-B)(n+1)(k-1)}{C_{k,\lambda}C_{k,\beta}\delta(n,k) - \lambda\beta(A-B)^2(n+1)^2}. \quad (2.16)$$

Since the function  $E(k)$  is an increasing function of  $k$  ( $k \geq 2$ ), letting  $k = 2$  in (2.16) we obtain

$$\zeta \leq E(2) = \frac{\lambda\beta(A-B)(1-B)}{C_{2,\lambda}C_{2,\beta} - \lambda\beta(A-B)^2(n+1)}, \quad (2.17)$$

which evidently proves Theorem 2.

Further, taking



$$f(z) = z - \frac{\lambda(A-B)}{C_{2,\lambda}} z^2 \quad (2.18)$$

and

$$g(z) = z - \frac{\beta(A-B)}{C_{2,\beta}} z^2 \quad (2.19)$$

we can show that the result of Theorem 2 is sharp.

**COROLLARY 2.** Let the functions  $f(z)$ ,  $g(z)$ ,  $h(z)$  be in the class  $T_{\lambda,n}^*(A,B)$ , then  $f * g * h(z) \in T_{\eta,n}^*(A,B)$ , where

$$\eta = \frac{\lambda^3(A-B)^2(1-B)}{C_{2,\lambda}^3 - \lambda^3(A-B)^3(n+1)}. \quad (2.20)$$

The result is best possible for the functions

$$f(z) = g(z) = h(z) = z - \frac{\lambda(A-B)}{C_{2,\lambda}} z^2. \quad (2.21)$$

**PROOF.** From Theorem 1, we have  $f * g(z) \in T_{\mu,n}^*(A,B)$ , where  $\mu$  is given by (2.1). We use now Theorem 2, we get  $f * g * h(z) \in T_{\eta,n}^*(A,B)$ , where

$$\eta = \frac{\lambda\mu(A-B)(1-B)}{C_{2,\lambda}C_{2,\mu} - \lambda\mu(A-B)^2(n+1)} = \frac{\lambda^3(A-B)^2(1-B)}{C_{2,\lambda}^3 - \lambda^3(A-B)^3(n+1)}.$$

This completes the proof of Corollary 2.

**THEOREM 3.** Let the functions  $f(z)$ ,  $g(z)$  defined by (1.1) and (1.2), respectively, be in the class  $T_{\lambda,n}^*(A,B)$ . Then the function  $h(z)$  defined by

$$h(z) = z - \sum_{k=2}^{\infty} (a_k^2 + b_k^2) z^k \quad (2.22)$$

belongs to the class  $T_{\varphi,n}^*(A,B)$ , where

$$\varphi = \frac{2\lambda^2(A-B)(1-B)}{C_{2,\lambda}^2 - 2\lambda^2(A-B)^2(n+1)}. \quad (2.23)$$

The result is sharp for the functions  $f(z) = g(z)$  defined by (2.13).

**PROOF.** By virtue of Lemma 1, we obtain

$$\sum_{k=2}^{\infty} \left\{ \frac{C_{k,\lambda} \delta(n,k)}{\lambda(A-B)(n+1)} \right\}^2 a_k^2 \leq \left\{ \sum_{k=2}^{\infty} \frac{C_{k,\lambda} \delta(n,k)}{\lambda(A-B)(n+1)} a_k \right\}^2 \leq 1 \quad (2.24)$$

and

$$\sum_{k=2}^{\infty} \left\{ \frac{C_{k,\lambda} \delta(n,k)}{\lambda(A-B)(n+1)} \right\}^2 b_k^2 \leq \left\{ \sum_{k=2}^{\infty} \frac{C_{k,\lambda} \delta(n,k)}{\lambda(A-B)(n+1)} b_k \right\}^2 \leq 1. \quad (2.25)$$

It follows from (2.24) and (2.25) that

$$\sum_{k=2}^{\infty} \frac{1}{2} \left\{ \frac{C_{k,\lambda} \delta(n,k)}{\lambda(A-B)(n+1)} \right\}^2 (a_k^2 + b_k^2) \leq 1. \quad (2.26)$$

Therefore, we need to find the largest  $\varphi$  such that

$$\frac{C_{k,\varphi} \delta(n,k)}{\varphi(A-B)(n+1)} \leq \frac{1}{2} \left\{ \frac{C_{k,\lambda} \delta(n,k)}{\lambda(A-B)(n+1)} \right\}^2 \quad (k \geq 2),$$

that is, that

$$\varphi \leq \frac{2(k-1)\lambda^2(A-B)(1-B)(n+1)}{C_{k,\lambda}^2 \delta(n,k) - 2\lambda^2(A-B)^2(n+1)^2}. \quad (2.27)$$

Since

$$G(k) = \frac{2(k-1)\lambda^2(A-B)(1-B)(n+1)}{C_{k,\lambda}^2 \delta(n,k) - 2\lambda^2(A-B)^2(n+1)^2} \quad (2.28)$$

is an increasing function of  $k$  ( $k \geq 2$ ), we readily have

$$\varphi \leq G(2) = \frac{2\lambda^2(A-B)(1-B)}{C_{2,\lambda}^2 - 2\lambda^2(A-B)^2(n+1)^2}, \quad (2.29)$$

which completes the proof of Theorem 3.

**THEOREM 4.** Let the function  $f(z) = z - \sum_{k=2}^{\infty} a_k z^k$  be in the class  $T_{\lambda,n}^*(A,B)$  and  $g(z) = z - \sum_{k=2}^{\infty} b_k z^k$  with  $|b_k| \leq 1$ ,  $k = 2, 3, \dots$ , then  $f * g(z) \in T_{\lambda,n}^*(A,B)$ .

**PROOF.** Since

$$\begin{aligned} \sum_{k=2}^{\infty} C_{k,\lambda} \delta(n,k) |a_k b_k| &= \sum_{k=2}^{\infty} C_{k,\lambda} \delta(n,k) a_k |b_k| \\ &\leq \sum_{k=2}^{\infty} C_{k,\lambda} \delta(n,k) a_k \leq \lambda(A-B)(n+1) \end{aligned}$$

by Lemma 1 it follows that  $f * g(z) \in T_{\lambda,n}^*(A,B)$ .

**COROLLARY 3.** If  $f(z) \in T_{\lambda,n}^*(A,B)$  and  $g(z) = z - \sum_{k=2}^{\infty} b_k z^k$  with  $0 \leq b_k \leq 1$ ,  $k = 2, 3, \dots$ , then  $f * g(z) \in T_{\lambda,n}^*(A,B)$ .

#### References

- [1] V. Kumar and A. M. Chaudhary, On a class of certain starlike functions with negative coefficients, Proc. Nat. Acad. Sci. India, 58 (A), II, (1988), 293-301.

- [2] A. Schild and H. Silverman, Convolution of univalent functions with negative coefficients, Ann. Univ. Mariae Curie-Sklodowska Sect. A, 29 (1975), 99-107.
- [3] St. Ruscheweyh, New criteria for univalent functions, Proc. Amer. Math. Soc. 49 (1975), 109-115.

M. K. Aouf

Department of Mathematics  
Faculty of Science  
University of Mansoura  
Mansoura, Egypt.

M. Nunokawa

Faculty of Education  
Gunma University  
4-2, Aramaki-machi, Maebashi  
Gunma 371, Japan.